Ranks of the Sylow 2-Subgroups of the Classical Groups

MONG LUNG LANG

Abstract. Let S be a 2-group. The rank (normal rank) of S is the maximal dimension of an elementary abelian subgroup (a normal elementary abelian subgroup) of S over $\mathbb{Z}_2$. The purpose of this article is to determine the rank and normal rank of S, where S is a Sylow 2-subgroup of the classical groups of odd characteristic.

1. Introduction

Sylow 2-subgroups of the classical groups (of odd characteristic) have been completely determined by Carter and Fong [CF] and Wong [W]. Let S be such a 2-group. The rank (normal rank) of S is the maximal dimension of an elementary abelian subgroup (a normal elementary abelian subgroup) of S over $\mathbb{Z}_2$. The purpose of this article is to determine the rank and normal rank of S. The result of this article is tabulated in the following table. The remaining of this article is organised as follows :

Section 2 gives some very basic facts about ranks and normal ranks of direct product and semidirect product of groups. Section 3 studies the ranks and normal ranks of wreath products. These results enable us to determine the ranks and normal ranks of Sylow 2-subgroups of $GL_n(q)$, $U_n(q)$, $O^+_{2n+1}(q)$, $O_{2n}(\eta, q)$, $SL_{2n+1}(q)$, $Sp_{2n}(q)$, and $\Omega_{2n}(\eta, q)$ ($\eta = \pm 1, q^n \equiv -\eta$) (mod 4). An alternative proof (cohomology free) of a lemma (Lemma 10.32 of [GLS2]) which is very useful in the study of p-groups is also provided.

Section 4 studies twisted wreath product of groups. Let $S(T, R, J)$ be a twisted wreath product (see section 4.2). We are able to give upper and lower bounds of $r_2(S(T, R, J))$. These bounds are by no means the best possible bounds for arbitrary T's and R's but will be proven in the section 5 that are optimal if S and T are the ones associated to the Sylow 2-subgroups of the classical groups (of odd characteristic). Our study of the twisted wreath product is motivated by the fact that if S is a Sylow 2-subgroup of the classical groups which cannot be described as a direct product of wreath products then S can be described by the usage of the twisted wreath product $S(T, R, J)$.

Section 5 gives the ranks and normal ranks of the Sylow 2-subgroups of classical groups that are described by twisted wreath product. They are easy consequences of Propositions 4.4 and 4.9 except for $\Omega_{2(2m+1)}(\eta, q)$ ($q^{2m+1} \equiv \eta$ (mod 4)) which requires some special treatment. Our analysis shows that certain invariance of TR determines uniquely the rank and normal rank of $S(T, R, J)$.

The notations we used are basically those in [CF] and [W]. In the following table, $\eta = \pm 1$, $\mathrm{ord}_2(q^2 - 1)$ is the largest integer m such that 2^m is a divisor of $q^2 - 1$ and that $[x]$ is the largest integer less than or equal to x.

2000 *Mathematics Subject Classification.* 20D06,20D08; Secondary 20D20.

Group	2 Rank	Normal 2 rank
$SL_{2n}(q)\ :\ q\equiv 3\,(4)$	$2n-1$	n
$SL_{2n}(q)\ :\ q\equiv 1\,(4)$	$2n-1$	$2n-1$
$SU_{2n}(q)\ :\ q\equiv 1\,(4)$	$2n-1$	n
$SU_{2n}(q)\ :\ q\equiv 3\,(4)$	$2n-1$	$2n-1$
$Sp_{2n}(q)$	n	n
$\Omega_{2n+1}(q)\ :\ \mathrm{ord}_2(q^2-1)\geq 4$	$2n$	n
$\Omega_{2n+1}(q)\ :\ \mathrm{ord}_2(q^2-1)=3$	$2n$	$2n$
$\Omega_{2n}(\eta,q)\ :\ q^n\equiv -\eta\,(4)\ \mathrm{ord}_2(q^2-1)\geq 4$	$2n-2$	$n-1$
$\Omega_{2n}(\eta,q)\ :\ q^n\equiv -\eta\,(4)\ \mathrm{ord}_2(q^2-1)=3$	$2n-2$	$2n-2$
$\Omega_{4n}(\eta,q)\ :\ q^n\equiv \eta\,(4)\ \mathrm{ord}_2(q^2-1)\geq 4$	$4n-1$	$2n$
$\Omega_{4n}(\eta,q)\ :\ q^n\equiv \eta\,(4)\ \mathrm{ord}_2(q^2-1)=3$	$4n-1$	$4n-1$
$\Omega_{4n+2}(\eta,q)\ :\ q^n\equiv \eta\,(4)\ \mathrm{ord}_2(q^2-1)\geq 4$	$4n+1$	$2n+1$
$\Omega_{4n+2}(\eta,q)\ :\ q^n\equiv \eta\,(4)\ \mathrm{ord}_2(q^2-1)=3$	$4n+1$	$4n+1$
$GL_n(q)\ :\ q\equiv 3\,(4)$	n	$[(n+1)/2]$
$GL_n(q)\ :\ q\equiv 1\,(4)$	n	n
$U_n(q)\ :\ q\equiv 1\,(4)$	n	$[(n+1)/2]$
$U_n(q)\ :\ q\equiv 3\,(4)$	n	n
$O^+_{2n+1}(q)\ :\ \mathrm{ord}_2(q^2-1)=3$	$2n$	$2n$
$O^+_{2n+1}(q)\ :\ \mathrm{ord}_2(q^2-1)\geq 4$	$2n$	n
$O_{2n}(\eta,q)\ :\ q^n\equiv \eta\,(4)\ \mathrm{ord}_2(q^2-1)=3$	$2n$	$2n$
$O_{2n}(\eta,q)\ :\ q^n\equiv \eta\,(4)\ \mathrm{ord}_2(q^2-1)\geq 4$	$2n$	n
$O_{2n}(\eta,q)\ :\ q^n\equiv -\eta\,(4)\ \mathrm{ord}_2(q^2-1)=3$	$2n$	$2n$
$O_{2n}(\eta,q)\ :\ q^n\equiv -\eta\,(4)\ \mathrm{ord}_2(q^2-1)\geq 4$	$2n$	$n+1$

2. Preliminaries : Ranks and Normal ranks of 2 groups

The main purpose of this section is to give some basic results that enable us to describe the ranks and normal ranks of 2-groups.

Definition 2.1. Lat p be a prime and let G be a group. We say G is of p-rank n if the p-rank of a Sylow p-subgroup of G is n. We say G is of normal p-rank n if the normal rank of a Sylow p-subgroup of G is n. The rank and normal rank of G are denoted by $r_p(G)$ and $nr_p(G)$ respectively. Recall that the rank (normal rank) of a p-group is the maximal dimension of an elementary abelian subgroup (a normal elementary abelian subgroup) over $\mathbb{Z}_p$.

Lemma 2.2. *Let $G = A\times B$. Then $r_p(G) = r_p(A) + r_p(B)$ and $nr_p(G) = nr_p(A) + nr_p(B)$.*

Proof. Let $E\subseteq G$ be elementary abelian of rank $r_p(G)$. Define the projection of E on A and B as follows.

$$E|A = \{a\in A\ :\ ab\in E \text{ for some } b\in B\},$$

$$E|B = \{b \in B \,:\, ab \in E \text{ for some } a \in A\}.$$

One sees easily that $E|A$ and $E|B$ are elementary abelian subgroups of A and B respectively. Further,

$$E \subseteq E|A \times E|B. \tag{2.1}$$

It follows easily from (2.1) that $r_p(G) = r_p(A) + r_p(B)$. Note that if E is normal, then both $E|A$ and $E|B$ are normal in A and B respectively as well. Hence $nr_p(G) = nr_p(A) + nr_p(B)$. □

Lemma 2.3. *Let $S = A \rtimes B$ be the semidirect product of A and B and let E be an elementary abelian subgroup of S. Suppose that $E = (E \cap A) \times R$. Then $r_2(E \cap A) \leq r_2(A)$, $r_2(R) \leq r_2(B)$, $r_2(R) = r_2(R|B)$. In particular, $r_2(S) \leq r_2(A) + r_2(B)$. In the case E is normal, the following holds. $r_2(E \cap A) \leq nr_2(A)$, $r_2(R) \leq r_2(B)$, and $nr_2(S) \leq nr_2(A) + r_2(B)$.*

Proof. Let E be an elementary abelian 2-group of S. Set $E = (E \cap A) \times R$. Since $E \cap A$ is in A, it is clear that

$$r_2(E \cap A) \leq r_2(A). \tag{2.2}$$

Note that elements in $R^\times$ are of the forms xb, where $x \in A, b \in B - \{1\}$. Let $M = \{x_1b_1, \cdots, x_nb_n\}$ be a minimal generating set of R. Since elements in M are of order 2, $o(b_i) = 2$ for all i. Since $x_ib_ix_jb_j = x_ix_j'b_ib_j$ is of order 2 as well, $o(b_ib_j) = 1$ or 2. This implies that $V = \langle b_1, b_2, \cdots, b_n\rangle$ is an elementary abelian 2-subgroup of B. Further $R|B = V$. Suppose that $\{b_1, b_2, \cdots, b_n\}$ is not a minimal generating set of V. Without loss generality, we may assume that $b_1 = \prod_{i=2}^n b_i^{e_i}$, where $e_i = 1$ or 2. It follows that

$$\prod_{i=2}^n (x_ib_i)^{e_i} = x_1'b_1.$$

Since M is a minimal generating set of R, $\langle x_1'b_1, x_1b_1\rangle = \langle x_1'b_1\rangle \times \langle x_1b_1\rangle \subseteq R$ is elementary abelian of order 4. Note that

$$\langle x_1'b_1\rangle \times \langle x_1b_1\rangle = \langle x_1'b_1\rangle \times \langle x_1b_1x_1'b_1\rangle \subseteq R.$$

As the group $\langle x_1b_1x_1'b_1\rangle$ (of order 2) is in $E \cap A$, we have just shown that R intersects $E \cap A$ nontrivially. A contradiction. Hence $\{b_1, b_2, \cdots, b_n\}$ is a minimal generating set of $V \subseteq B$. It follows that $r_2(R) = |M| = n = r_2(V) = r_2(R|B)$ and that $r_2(R) = n \leq r_2(B)$. As a consequence, $r_2(E) = r_2(E \cap A) + r_2(R) \leq r_2(A) + r_2(B)$.

In the case E is normal, $E \cap A$ is a normal subgroup of A. Hence $r_2(E \cap A) \leq nr_2(A)$. This completes the proof of the lemma. □

3. Wreath Product

We shall first give a (cohomology free) proof of the following fact which is very useful in our study of the normal ranks of p-groups. The purpose of such proof is to keep our study of the 2-ranks elementary. Our main results concerning the ranks and normal ranks can be found in Proposition 3.4.

Lemma 3.1. (Lemma 10.32 of [GLS2]) *Let $P = Q_1 \times Q_2 \times \cdots \times Q_p$, where the Q_i are cycled by the element x of order p (whence $P\langle x\rangle \cong Q_1 \wr \mathbb{Z}_p$, the wreath product of Q_1 and $\mathbb{Z}_p$). Then every element of order p in the coset xP is a P-conjugate of x.*

Proof. We note first that every P-conjugate pxp^{-1} are members in xP. This follows from the fact that P is a normal subgroup of $P\langle x\rangle$.

Note also that $q_1q_2\cdots q_p \in C_P(x)$ ($q_i \in Q_i$) if and only if $xq_1q_2\cdots q_px^{-1} = q_1q_2\cdots q_p$. As P is the direct product of the Q_i's, we must have

$$xq_px^{-1} = q_1, xq_ix^{-1} = q_{i+1}, \text{ where } i = 1, 2, \cdots, p-1.$$

As a consequence, there are $|Q_1|$ choices of q_1 and the remaining q_i's ($i \geq 2$) are uniquely determined by q_1 and x. Hence

$$|C_P(x)| = |Q_1|.$$

This implies that x possesses $|P|/|C_P(x)| = |Q_1|^{p-1}$ P-conjugates in xP. Note that such conjugates are elements of order p.

In order to complete our lemma, it suffices to prove that Px has $|Q_1|^{p-1}$ elements of order p. Note that $q_1q_2\cdots q_px$ is of order p if and only if $(q_1q_2\cdots q_px)^p = 1$. It follows that

$$(q_1q_2\cdots q_p)(q_1q_2\cdots q_p)^x(q_1q_2\cdots q_p)^{x^2}\cdots(q_1q_2\cdots q_p)^{x^{p-1}} = 1,$$

where $q^x = xqx^{-1}$, $q_1, q_p^x, q_{p-1}^{x^2}, \cdots, q_2^{x^{p-1}} \in Q_1$. Since the product $\prod Q_i$ is a direct product, we must have

$$q_1q_p^xq_{p-1}^{x^2}\cdots q_2^{x^{p-1}} = 1.$$

It follows that the choices of q_1 is uniquely determines by x and $q_2, q_3, \cdots, q_{p-1}$. Hence xP possesses at most $|Q_1|^{p-1}$ elements of order p. This completes the proof of our lemma. □

Remark. (i) Denote by $d(X)$ the number of elements of order p of X. Our lemma implies that x^iP ($1 \leq i \leq p-1$) possesses $|Q_1|^{p-1}$ elements of order p. It is clear that P possesses $(d(Q_1)+1)^p - 1$ elements of order p. As a consequence,

$$d(P\langle x\rangle) = (d(Q_1)+1)^p - 1 + (p-1)|Q_1|^{p-1}.$$

(ii) In the case Q_1 is a Sylow p-subgroup of S_{p^n}, the cycle decomposition of x is p^{n+1} and $P\langle x\rangle$ is a Sylow p-subgroup of $S_{p^{n+1}}$. Denote by v_{n+1} the number of elements of $P\langle x\rangle$ with cycle decomposition p^{n+1}. Our lemma implies that x^iP ($1 \leq i \leq p-1$) possesses $|Q_1|^{p-1}$ elements with cycle decomposition p^{n+1}. It is clear that P possesses v_n^p elements with cycle decomposition p^{n+1}. As a consequence, we have

$$v_{n+1} = v_n^p + (p-1)|Q_1|^{p-1}. \tag{3.1}$$

Lemma 3.2. *Let $S = (Q_1 \times Q_2) \rtimes \langle x\rangle \cong Q_1 \wr \mathbb{Z}_2$ be a 2-group. Then $r_2(S) = 2\cdot r_2(Q_1)$. Suppose that the rank of Q_1 is at least 2. Then every elementary abelian subgroup of dimension $r_2(S)$ is contained in $Q_1 \times Q_2$.*

Proof. Let E be an elementary abelian subgroup of S of dimension $r_2(S)$. Set $E = A \times B$, where $A = E \cap (Q_1 \times Q_2)$. Applying Lemma 2.3, $r_2(B) \leq 1$. In the case $r_2(B) = 0$, E is a subgroup of $Q_1 \times Q_2$. The maximality of E (in dimension) implies that $r_2(E) = 2\cdot r_2(Q_1)$. Hence $r_2(S) = r_2(E) = 2\cdot r_2(Q_1)$. In the case $r_2(B) = 1$, B is generated by an element of the form $y = q_1q_2x$, where $q_i \in Q_i$. Note that $Q_1 \cap Q_2 = 1$, $[Q_1, Q_2] = 1$ and that the conjugation action of y interchanges Q_1 and Q_2. Let $t_1t_2 \in E \cap (Q_1 \times Q_2)$, where $t_i \in Q_i$. Since E is abelian,

$$[y, t_1t_2] = 1. \tag{3.2}$$

It follows that

$$yt_1y^{-1} = t_2 \text{ and } y^{-1}t_2y = t_1. \tag{3.3}$$

Hence t_2 (of order 2) is uniquely determined by t_1 (of order 2) and y. As a consequence, $r_2(E \cap (Q_1 \times Q_2)) \leq r_2(Q_1)$. It follows that

$$r_2(S) = r_2(E) \leq 1 + r_2(Q_1). \tag{3.4}$$

Since $Q_1 \cong Q_2$ (x permutes Q_1 and Q_2), the rank of $Q_1 \times Q_2$ is $2 \cdot r_2(Q_1)$. This together with (3.4) gives

$$2 \cdot r_2(Q_1) \leq r_2(S) \leq 1 + r_2(Q_1).$$

Hence

$$r_2(Q_1) = r_2(Q_2) = 1. \tag{3.5}$$

It follows that $r_2(S) = 2 = 2 \cdot r_2(Q)$. This completes the proof of the first part of our lemma.

Suppose that S possesses an elementary abelian subgroup E such that E is not a subgroup of $Q_1 \times Q_2$. It follows from the above that $r_2(Q_1) = 1$. A contradiction. Hence $E \subseteq Q_1 \times Q_2$. □

Remark. Suppose that $E \subseteq S$ is elementary abelian of dimension $2r_2(Q_1) - 1$, where $r(Q_1) \geq 2$. Applying the proof of our lemma (3.2), one can show that E is a subgroup of $Q_1 \times Q_2$ as well.

Lemma 3.3. *Let S be a 2-group of the form $(Q_1 \times Q_2) \rtimes \langle x \rangle \cong Q_1 \wr \mathbb{Z}_2$. Then $nr_2(S) = 2 \cdot nr_2(Q_1)$. Further, the following hold.*

(i) *Suppose that Q_1 is not elementary abelian. Then every normal elementary abelian subgroup (not necessarily of maximal dimension) of S is contained in $Q_1 \times Q_2$.*

(ii) *Suppose that that Q_1 is not isomorphic to $\mathbb{Z}_2$. Then every normal elementary abelian subgroup of dimension $r_2(S)$ is contained in $Q_1 \times Q_2$.*

Proof. Let $E \subseteq S$ be a normal elementary abelian 2-group of dimension $nr_2(S)$. If $E \subseteq Q_1 \times Q_2$, applying Lemma 2.2, we have $nr_2(S) = 2nr_2(Q_1)$. We shall therefore assume that E is not a subgroup of $Q_1 \times Q_2$. It follows that E has an element of order 2 of the form $q_1 q_2 x$. Applying Lemma 3.1, E contains x. Note that $Q_1 \cap Q_2 = 1$, $[Q_1, Q_2] = 1$ and that the conjugation action of x interchanges Q_1 and Q_2. Hence $q_1 q_2 x$ is of order 2 if and only if

$$q_1 x q_2 x = 1. \tag{3.6}$$

Since E is abelian and $q_1 q_2 x, x \in E$, we have $x q_1 q_2 x = q_1 q_2 x^2 = q_1 q_2$. Hence

$$q_1 = x q_2 x. \tag{3.7}$$

Applying (3.6) and (3.7), we have $q_1 = q_1^{-1}$. Hence the order of q_1 is either 1 or 2. Since this is true for all $q_1 \in Q_1$ (for any $q_1 \in Q_1$, let $q_2 = x q_1^{-1} x$, then equation (3.6) holds), we conclude that Q_1 is an elementary abelian 2-group. It is now clear that $nr_2(S) = 2nr_2(Q_1)$.

(i) Suppose that S possesses a normal elementary abelian subgroup F such that F is not a subgroup of $Q_1 \times Q_2$. It follows from the above that Q_1 is elementary abelian. A contradiction.

(ii) Suppose that S possesses a normal elementary abelian subgroup E of dimension $nr_2(S)$ such that E is not a subgroup of $Q_1 \times Q_2$. It follows from the above that Q_1 is elementary abelian. Set

$$E = A \times B, \text{ where } A = E \cap (Q_1 \times Q_2). \tag{3.8}$$

Applying the proof of Lemma 3.2 (equation (3.5)), the rank of Q_1 is 1. It follows that $Q_1 \cong \mathbb{Z}_2$. A contradiction. Hence $E \subseteq Q_1 \times Q_2$. □

Proposition 3.4. *Let T be a 2-group and let $w_n(T)$ be the n-th wreath product of T and $\mathbb{Z}_2$ ($w_1(T) = T \wr \mathbb{Z}_2, \cdots, w_n(T) = w_{n-1}(T) \wr \mathbb{Z}_2$). Then $r_2(w_n(T)) = 2^n r_2(T)$ and $nr_2(w_n(T)) = 2^n nr_2(T)$.*

Proof. Since $r_2(w_k(T)) \geq 2$ and $w_k(T)$ is not elementary abelian whenever $k \geq 1$, we may apply the second part of Lemma 3.2 and (i) of Lemma 3.3 to conclude that $r_2(w_n(T)) = 2^{n-1} r_2(w_1(T))$ and $nr_2(w_n(T)) = 2^{n-1} nr_2(w_1(T))$. Applying the first part of Lemma 3.2, we have $r_2(w_1(T) = 2r_2(T)$. Applying the first part of Lemma 3.3, we have $nr_2(w_1(T) = 2nr_2(T)$. This completes the proof of the proposition. □

3.1. **Ranks and normal ranks of Sylow 2-subgroups of $GL_n(q)$, $U_n(q)$, $O^+_{2n+1}(q)$, $O_{2n}(\eta, q)$.** Sylow 2-subgroups of the above groups are determined by Carter and Fong [CF]. They are described as the direct product of certain wreath products. Their rank and normal rank can be determined by applying Proposition 3.4. The result of our study can be found in the introduction.

4. Twisted wreath product

Definition 4.1. (Twisted wreath product) Let $T \rtimes R$ be a semidirect product of 2-groups, where T is normal. Define

$$w_1(T, R, J) = \left\langle \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,:\, r \in R \right\rangle.$$

The above group is called the first twisted wreath product of T and R ($w_0(T, R, J) = T$), where

$$J = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}_2.$$

Define similarly the second twisted wreath product $w_2(T, R, J)$ to be the following group :

$$\left\langle \begin{pmatrix} w_1(T,R,J) & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & w_1(T,R,J) \end{pmatrix}, \begin{pmatrix} d_2(r) & 0 \\ 0 & d_2(r)^{-1} \end{pmatrix}, \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \right\rangle,$$

where $d_2(r)$ is the 2×2 diagonal matrix diag $(r, 1)$, $r \in R$. One may define inductively $w_{n+1}(T, R, J)$ to be the following (see [W] for more detail) :

$$\left\langle \begin{pmatrix} w_n(T,R,J) & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & w_n(T,R,J) \end{pmatrix}, \begin{pmatrix} d_n(r) & 0 \\ 0 & d_n(r)^{-1} \end{pmatrix}, \begin{pmatrix} 0 & I_{2^n} \\ I_{2^n} & 0 \end{pmatrix} \right\rangle,$$

where $d_n(r)$ is the $2^n \times 2^n$ diagonal matrix diag $(r, 1, \cdots, 1)$, $r \in R$. Define

$$R_{n+1} = \left\langle \operatorname{diag}\left(d_n(r), d_n(r)^{-1}\right) \right\rangle, \quad D_{n+1} = \langle d_{n+1}(r) \,:\, r \in R \rangle. \tag{4.1}$$

4.1. **Subgroups of $w_{n+1}(T, R, J)$.** We shall define some subgroups (inductively) of $w_{n+1}(T, R, J)$ as follows :

(i) $w_0((T, 1, J) = T$, and $w_{n+1}(T, 1, J)$ is defined to be

$$\left\langle \begin{pmatrix} w_n(T,1,J) & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & w_n(T,1,J) \end{pmatrix}, \begin{pmatrix} 0 & I_{2^n} \\ I_{2^n} & 0 \end{pmatrix} \right\rangle. \tag{4.2}$$

(ii) $w_0(T, R, 1) = T$, and $w_{n+1}(T, R, 1)$ is defined to be

$$\left\langle \begin{pmatrix} w_n(T,R,1) & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & w_n(T,R,1) \end{pmatrix}, \begin{pmatrix} d_n(r) & 0 \\ 0 & d_n(r)^{-1} \end{pmatrix} \right\rangle. \tag{4.3}$$

(iii) $w_0(1,1,J)=1$, and $w_{n+1}(1,1,J)$ is defined to be

$$\left\langle \begin{pmatrix} w_n(1,1,J) & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & w_n(1,1,J) \end{pmatrix}, \begin{pmatrix} 0 & I_{2^n} \\ I_{2^n} & 0 \end{pmatrix} \right\rangle. \tag{4.4}$$

(iv) $w_0(T,1,1)=T$, and $w_{n+1}(T,1,1)$ is defined to be

$$\left\langle \begin{pmatrix} w_n(T,1,1) & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & w_n(T,1,1) \end{pmatrix} \right\rangle. \tag{4.5}$$

It is clear that

(v) $w_n(T,1,J)$ is just the n-th wreath product of T and $J \cong \mathbb{Z}_2$. Equivalently, $w_n(T,1,J) \cong w_n(T)$.
(vi) $w_n(T,R,1)$ is a normal subgroup of $w_n(T,R,J)$.
(vii) $w_n(1,1,J)$ consists of permutation matrices only. It is isomorphic to a Sylow 2-subgroup of S_{2^n}.
(viii) $w_n(T,1,1)$ is a direct product of 2^n copies of T's and $w_n(T,1,1)$ is a normal subgroup of $w_n(T,R,J)$.

It is also clear that

$$w_n(T,R,J) \subseteq w_n(T,R,J)D_n = w_n(TR,1,J). \tag{4.6}$$

4.2. **Sylow 2-subgroups of the classical groups.** The main purpose of this section is the construction of the group we define in (4.7). We will see later (in section 5) that Sylow 2-subgroups of the classical groups can be described by (4.7). Let $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_u}$ $(m_1 < m_2 < \cdots)$ be the 2-adic representation of n. In the case $u = 1$, we define $U(R) = 1$. In the case $u \geq 2$, define $U(R)$ to be the group generated by

$$\begin{gathered}
\operatorname{diag}(d_{m_1}(r), d_{m_2}(r), I_{2^{m_3}}, I_{2^{m_4}}, \cdots, I_{2^{m_u}}), \\
\operatorname{diag}(I_{2^{m_1}}, d_{m_2}(r), d_{m_3}(r), I_{2^{m_4}}, \cdots, I_{2^{m_u}}), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\operatorname{diag}(I_{2^{m_1}}, \cdots, I_{2^{m_{u-2}}}, d_{m_{u-1}}(r), d_{m_u}(r)),
\end{gathered}$$

where $r \in R$. It is clear that $U(R)$ normalises $W(T,R,J)$, where $W(T,R,J)$ is defined to be the following group.

$$W(T,R,J) = \operatorname{diag}(w_{m_1}(T,R,J), w_{m_2}(T,R,J), \cdots, w_{m_u}(T,R,J)).$$

Define

$$S(T,R,J) = W(T,R,J) \rtimes U(R). \tag{4.7}$$

Note that we may define similarly the groups $W(T,R,1)$, $W(1,1,J)$, $S(T,R,1)$ to be

$$\begin{aligned}
W(T,R,1) &= \operatorname{diag}(w_{m_1}(T,R,1), w_{m_2}(T,R,1), \cdots, w_{m_u}(T,R,1)), \\
W(1,1,J) &= \operatorname{diag}(w_{m_1}(1,1,J), w_{m_2}(1,1,J), \cdots, w_{m_u}(1,1,J)),
\end{aligned}$$

and

$$S(T,R,1) = W(T,R,1) \rtimes U(R)$$

respectively. It follows from (4.6) that $S(T,R,J) \subseteq S(TR,1,J) = W(TR,1,J)$. By (v) of the above, one has

$$W(TR,1,J) \cong \prod_{i=1}^{u} w_{m_i}(TR). \tag{4.8}$$

Remark. An easy observation of the elements in R_n (see (4.1)) and $U(R)$ shows that an element diag $(x_1, x_2, \cdots) \in W(TR, 1, J)$ is in $S(T, R, J)$ only if the numbers of the i's such that $x_i = t_i r_i$ ($t_i \in T$, $r_i \in R$, $r_i \neq 1$) is even.

4.3. **Normal elementary abelian subgroups of** $S(T, R, J)$**.** Let E be a normal elementary abelian subgroup of $S(T, R, J)$ of dimension $nr_2(S(T, R, J))$ The purpose of this section is to give a supergroup of E (Lemma 4.2) and a lower bound of $r_2(E)$ (Lemma 4.3). Proposition 4.4 is an easy consequence of Lemmas 4.2 and 4.3 which works well for the Sylow 2-subgroups of the classical groups.

Lemma 4.2. *Suppose that T is not elementary abelian. Let E be a normal elementary abelian subgroup of $S(T, R, J)$. Then $E \subseteq S(T, R, 1) = W(T, R, 1) \rtimes U(R)$. In particular, $r_2(E) \leq n \cdot r_2(T) + (n-1)r_2(R)$.*

Proof. Suppose not. Then E has an element of the form $wu\sigma$, where $w \in W(T, R, 1)$, $u \in U(R)$ and $\sigma \in W(1, 1, J) - \{1\}$. Since the order of $wu\sigma$ is 2, σ is of order 2. It is easy to see that $\tau = wu\sigma$ normalises $S(T, 1, 1)$. Note that $S(T, 1, 1)$ is the direct product of n copies of T and that the conjugation of τ permutes these T's with a cycle decomposition 2^m for some $m \geq 1$. Without loss of generality, we may assume that τ interchanges the first two copies of the T's. Label them as Q_1 and Q_2, one has

$$\tau \in N = E \cap (Q_1 \times Q_2) \rtimes \langle \tau \rangle \,.$$

Since N is a normal subgroup of $(Q_1 \times Q_2) \rtimes \langle \tau \rangle$, we may apply (i) of Lemma 3.3 to conclude that $N \subseteq Q_1 \times Q_2$. A contradiction. Hence $E \subseteq W(T, R, 1) \rtimes U(R)$. Applying Lemma 2.3, it is clear that

$$r_2(E) \leq r_2(W(T, R, 1) \rtimes U(R)) \leq r_2(W(T, R, 1)) + r_2(U(R)).$$

By Lemma 2.3,

$$r_2(W(T, R, 1)) \leq \sum_{i=1}^{u} (2^{m_i} r_2(T) + (2^{m_i} - 1) r_2(R)) \leq n \cdot r_2(T) + (n-u) r_2(R).$$

Note also that $r_2((U(R)) = (u-1)r_2(R)$. As a consequence,

$$r_2(E) \leq n \cdot r_2(T) + (n-1) r_2(R). \qquad \square$$

Let E_0 and R_0 be elementary abelian subgroups of T and R respectively such that $\langle E_0, R_0 \rangle = E_0 \times R_0$ is elementary abelian normal in TR. For $t \in T$, $r \in R_0$, Since $\langle E_0, R_0 \rangle$ is elementary abelian normal in TR, $t^{-1} r t r^{-1} \in E_0 \times R_0$. On the other hand, since R_0 normalises T, $t^{-1} r t r^{-1} \in T$. Hence $t^{-1} r t r^{-1} \in (E_0 \times R_0) \cap T = E_0$. Hence

$$[T, R_0] \subseteq E_0. \tag{4.9}$$

Note also that R_0 is normal in R (R is abelian) and that the normality of $E_0 \times R_0$ in TR implies that $E_0 \lhd T$. Hence

$$R_0 \lhd R\,,\ E_0 \lhd T. \tag{4.10}$$

Lemma 4.3. *Let E_0 and R_0 be given as in the above and let E be a normal elementary abelian subgroup of $S(T, R, J)$ of dimension $nr_2(S(T, R, J))$. Then $r_2(E) \geq n \cdot r_2(E_0) + (n-1) r_2(R_0)$.*

Proof. Let $N = S(E_0, R_0, 1)$. Since $[E_0, R_0] = 1$, N is elementary abelian. Applying (4.9) and (4.10), one can show that N is a normal subgroup of $S(T, R, J)$.

$$r_2(N) = r_2(W(E_0, R_0, 1) \times U(R_0)) = r_2(W(E_0, R_0, 1)) + r_2(U(R_0))$$

It is easy to see that

$$r_2(W(E_0,R_0,1)) = \sum_{i=1}^{u}(2^{m_i}r_2(E_0)+(2^{m_i}-1)r_2(R_0)) = n\cdot r_2(E_0)+(n-u)r_2(R_0).$$

Note that $r_2((U(R_0)) = (u-1)r_2(R_0)$. This implies that $r_2(N) = n\cdot r_2(E_0) + (n-1)r_2(R_0)$. As a consequence,

$$r_2(E) \geq r_2(N) = n\cdot r_2(E_0) + (n-1)r_2(R_0). \qquad \square$$

Applying Lemmas 4.2 and 4.3, we have the following proposition. Note that this proposition has its importance as many Sylow 2-subgroups of the classical groups satisfy the assumptions of this proposition.

Proposition 4.4. *Let T and R be given as in Definition 4.1. Suppose that T is not elementary abelian. Then the following hold.*

- (i) *Suppose that there exists $E_0 \lhd T$, $R_0 \lhd E$ (E_0 and R_0 are elementary abelian with $\langle E_0, R_0\rangle = E_0\times R_0$) such that $r_2(T) = r_2(E_0)$, $r_2(R) = r_2(R_0)$. Then the normal rank of $S(T,R,J)$ is $n\cdot r_2(T)+(n-1)r_2(R)$.*
- (ii) *Suppose that for every involution $x \in TR - T$, there exists some $t\in T$ such that $\langle x, txt^{-1}\rangle$ is not abelian. Let E be a normal elementary abelian subgroup of $S(T,R,J)$. Then $E \subseteq S(T,1,1) = W(T,1,1)$. In particular, $r_2(E) = n\cdot nr_2(M)$, where M is the largest (in dimension) elementary abelian normal subgroup of TR that contains in T.*

Proof. (i) is an immediate consequence of Lemmas 4.2 and 4.3. Suppose that TR satisfies the assumption of (ii). Then normal elementary abelian subgroups of TR are subgroups of T. By Lemma 4.2, $E \subseteq S(T,R,1) = W(T,R,1) \rtimes U(R)$ (note that members in $S(T,R,1)$ are of the forms $\mathrm{diag}\,(\cdots\cdots))$. Let

$$u = \mathrm{diag}\,(e, e_2, \cdots, e_n) \in E.$$

Suppose that $e \notin T$. By our assumption, there exists $x\in T$ such that $\langle e, txt^{-1}\rangle$ is not abelian. Let

$$v = \mathrm{diag}\,(x, x_2, \cdots, x_n) \in W(T,R,1)\rtimes U(R) \subseteq S(T,R,J).$$

Then $\langle u, vuv^{-1}\rangle \subseteq E$ is not abelian. A contradiction. Hence $e\in T$. Repeat this argument for all the e_i's, we conclude that $E\subseteq S(T,1,1)$. It is now an easy matter to see that $r_2(E) = n\cdot nr_2(M)$. $\square$

4.4. **Elementary abelian subgroups of $S(T,R,J)$.** Let E be an elementary abelian subgroup of $S(T,R,J)$ of dimension $r_2(S(T,R,J))$. The purpose of this section is to give upper and lower bounds of $r_2(E)$ (Lemmas 4.5-4.8). Proposition 4.9 is an easy consequence of Lemmas 4.5-4.8 which works well for the Sylow 2-subgroups of the classical groups.

Let E_0 and R_0 be elementary abelian subgroups of T and R respectively such that $\langle E_0, R_0\rangle = E_0\times R_0$ is elementary abelian in TR. Similar to Lemma 4.3, we have the following :

Lemma 4.5. *Let E be an elementary abelian subgroup of $S(T,R,J)$ of dimension $r_2(S(T,R,J))$. Then $r_2(E)\geq n\cdot r_2(E_0)+(n-1)r_2(R_0)$.*

Lemma 4.6. $r_2(S(T,R,J)) \leq n\cdot r_2(T) + (n-1)r_2(R)$.

Proof. By Lemma 2.3,

$$r_2(S(T,R,J)) \leq r_2(W(T,R,J)) + r_2(U(R)) = \sum r_2(w_{m_i}(T,R,J)) + (u-1)r_2(R).$$

By Lemma 2.3 and the definition of $w_{n+1}(T,R,J)$, we have $r_2(w_{n+1}(T,R,J)) \leq 2r_2(w_n(T,R,J)) + r_2(R) + 1$. Suppose that

$$r_2(w_{n+1}(T,R,J)) = 2r_2(w_n(T,R,J)) + r_2(R) + 1. \tag{4.11}$$

Let E be elementary abelian of dimension $r_2(S(T,R,J))$. It follows from the equality (4.11) that E contains an element of the form $\tau = g\begin{pmatrix} 0 & I_{2^n} \\ I_{2^n} & 0 \end{pmatrix}$. Consider the conjugation action of τ on

$$F = E \cap \left\langle Q_1 = \begin{pmatrix} w_n(T,R,J) & 0 \\ 0 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 1 & 0 \\ 0 & w_n(T,R,J) \end{pmatrix} \right\rangle.$$

Let xy ($x \in Q_1$, $y \in Q_2$) be an element of F. Since F is abelian, one has

$$xy = \tau^{-1}xy\tau = (\tau^{-1}x\tau)(\tau^{-1}y\tau).$$

Since the action of τ interchanges Q_1 and Q_2 and $Q_1 \cap Q_2 = 1$, we have $x = \tau^{-1}y\tau$. As a consequence, y is determined uniquely by x. Hence the rank of F is at most $r_2(w_n(T,R,J))$. On the other hand, (4.11) and Lemma 2.3 imply that $r_2(F) = 2r_2(w_n(T,R,J))$. A contradiction. Hence (4.11) is false. It follows that

$$r_2(w_{n+1}(T,R,J)) \leq 2r_2(w_n(T,R,J)) + r_2(R). \tag{4.12}$$

Hence

$$\sum r_2(w_{m_i}(T,R,J)) \leq \sum (2^{m_i} r_2(T) + (2^{m_i} - 1)r_2(R) = n \cdot r_2(T) + (n-u)r_2(R).$$

As a consequence,

$$r_2(S(T,R,J)) \leq n \cdot r_2(T) + (n-1)r_2(R). \qquad \square$$

Lemma 4.7. $r_2(S(T,R,J)) \leq n \cdot r_2(TR)$. *Suppose that* $r_2(T) = r_2(TR)$. *Then* $r_2(S(T,R,J)) = n \cdot r_2(T)$.

Proof. Since $w_n(TR,1,J)$ is a wreath product (see (i) of section 4.1)), we may apply Proposition 3.4 and conclude that

$$r_2(w_n(TR,1,J)) = 2^n r_2(TR). \tag{4.13}$$

By (4.8),

$$S(T,R,J) \subseteq S(TR,1,J) = W(TR,1,J). \tag{4.14}$$

By (4.13) and (4.14), we have

$$r_2(S(T,R,J)) \leq r_2(W(TR,1,J)) = \sum 2^{m_i} r_2(TR) = n \cdot r_2(TR).$$

It is clear that

$$r_2(S(T,R,J)) \geq r_2(S(T,1,1)) = n \cdot r_2(T).$$

This completes the proof of the lemma. $\square$

Lemma 4.8. *Suppose that* $r_2(TR) = r_2(T) + 1 \geq 3$. *Suppose further that there exists* $E_0 \subseteq T$, $R_0 \subseteq E$ *(*E_0 *and* R_0 *are elementary abelian) such that* $[E_0, R_0] = 1$, $r_2(E_0) = r_2(T)$, $r_2(R_0) = 1$. *Then* $r_2(S(T,R,J)) = n \cdot r_2(TR) - 1$.

Proof. Since E_0, R_0 are elementary abelian with $[E_0, R_0] = 1$, it follows that $r_2(S(E_0,R_0,1)) = n \cdot r_2(E_0) + (n-1)r_2(R_0)$. Since $r_2(TR) = r_2(T) + 1$, $r_2(E_0) = r_2(T)$, $r_2(R_0) = 1$, we have

$$r_2(S(T,R,J)) \geq n \cdot r_2(E_0) + (n-1)r_2(R_0) = n \cdot r_2(TR) - 1. \tag{4.15}$$

By Lemma 3.2, $r_2(W(TR,1,J)) = n \cdot r_2(TR)$ and an elementary abelian subgroup of $W(TR,1,J)$ of dimension $r_2(W(TR,1,J))$ is a subgroup of $W(TR,1,1)$. As a

consequence, if E is elementary abelian subgroup of dimension $r_2(W(TR,1,J))$, then

$$E = \mathrm{diag}\,(E_1, E_2, \cdots, E_n),$$

where $r_2(E_i) = r_2(TR)$ and the E_i's are permuted by $W(1,1,J)$. Since $r_2(TR) = r_2(T) + 1 > r_2(T)$, E_1 is not a subgroup of T. Hence E_1 must admit an element of the form tr, where $r \in R - \{1\}$. Note that

$$\mathrm{diag}\,(tr, 1 \cdots, 1) \in E$$

is not in $S(T,R,J)$ (see remark of section 4.2). As a consequence, the rank of $S(T,R,J)$ is less than $r_2(E) = n \cdot r_2(TR)$. Applying (4.15), we may now conclude that $r_2(S(T,R,J)) = n \cdot r_2(TR) - 1$. $\square$

The following is a consequence of Lemmas 4.5-4.8. It is useful in the study of the ranks of Sylow 2-subgroups of the classical groups.

Proposition 4.9. *The following hold.*

(i) *Suppose that there exists $E_0 \subseteq T$, $R_0 \subseteq R$ (E_0 and R_0 are elementary abelian) such that $[E_0, R_0] = 1$, $r_2(E_0) = r_2(T)$, $r_2(R_0) = r_2(R)$. Then*

$$r_2(S(T,R,J)) = n \cdot r_2(T) + (n-1)r_2(R).$$

(ii) *Suppose that $r_2(TR) = r_2(T)$. Then $r_2(S(T,R,J)) = n \cdot r_2(T)$.*

(iii) *Suppose that $r_2(TR) = r_2(T) + 1 \geq 3$. Suppose further that there exists $E_0 \subseteq T$, $R_0 \subseteq R$ (E_0 and R_0 are elementary abelian) such that $[E_0, R_0] = 1$, $r_2(E_0) = r_2(T)$, $r_2(R_0) = 1$. Then $r_2(S(T,R,J)) = n \cdot r_2(TR) - 1$.*

5. Applications : Ranks and Normal ranks of Groups of Lie type

5.1. **Special linear and special unitary groups I.** Let 2^{t+1} be the largest power of 2 that divides $q^2 - 1$ and let T be a generalised quaternion group $\langle v, w \rangle$, where $o(v) = 2^t, o(w) = 4$. Suppose that T is normalised by $R = \langle e \rangle$ (a group of order 2) :

$$eve^{-1} = v^{-1}\,,\; ewe^{-1} = vw.$$

Let $n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_u}$ be the 2-adic representation of n. By Theorem 4 of [W], $S(T,R,J)$ is a Sylow 2-subgroup of $SL(2n,q)$ ($q \equiv 3 \pmod 4$) and $SU(2n,q)$ ($q \equiv 1 \pmod 4$).

(a) **Normal Rank of** $S(T,R,J)$**.** One can show easily that $x \in TR - T$ is an involution if x is of the form $v^i e$. Further, $\left\langle x, wxw^{-1} \right\rangle$ is not abelian. Applying (ii) of Proposition 4.4,

$$nr_2(S(T,R,J)) = n \cdot r_2(T) = n.$$

(b) **Rank of** $S(T,R,J)$**.** One sees easily that $r_2(TR) = 2$, $r_2(T) = r_2(R) = 1$. Further, $E_0 = \left\langle v^{2^{t-1}} \right\rangle$ and $R_0 = \langle e \rangle$ satisfy the assumption of (i) of Proposition 4.9. Hence

$$r_2(S(T,R,J)) = 2n - 1.$$

Remark. In the case S is a Sylow 2-subgroup of $SL(2n+1,q)$ ($q \equiv 3 \pmod 4$) or $SU(2n+1,q)$ ($q \equiv 1 \pmod 4$) it is well known that S is isomorphic to a Sylow 2-subgroup of $GL(2n,q)$. Its rank and normal rank are given in section 3.1.

5.2. **Special linear and special unitary groups II.** Let 2^{t+1} be the largest power of 2 that divides q^2-1 and let T be a generalised quaternion group $\langle v,w\rangle$, where $o(v)=2^t, o(w)=4$. Suppose that T is normalised by e (an element of order 2^t) :

$$eve^{-1}=v\,,\ ewe^{-1}=vw.$$

Let $n=2^{m_1}+2^{m_2}+\cdots+2^{m_u}$ be the 2-adic representation of n. Define $R=\langle e\rangle$. By Theorem 4 of [W], $S(T,R,J)$ is a Sylow 2-subgroup of $SL(2n,q)$ ($q\equiv 1 \pmod 4$) and $SU(2n,q)$ ($q\equiv 3 \pmod 4$). Let $E_0=\left\langle w^2=v^{2^{t-1}},\right\rangle \lhd T$, $R_0=\left\langle f=e^{2^{t-1}}\right\rangle \lhd R$. Then $[E_0,R_0]=1$ and $E_0\times R_0$ is normal elementary abelian in TR.

(a) **Normal Rank of** $S(T,R,J)$**.** Since $r_2(T)=r_2(E_0)$, $r_2(R)=r_2(R_0)$, one may apply (i) of Proposition 4.4 and conclude that

$$nr_2(S(T,R,J))=2n-1.$$

(b) **Rank of** $S(T,R,J)$**.** By (i) of Proposition 4.9, we have

$$r_2(S(T,R,J))=2n-1.$$

Remark. In the case S is a Sylow 2-subgroup of $SL(2n+1,q)$ ($q\equiv 1 \pmod 4$) or $SU(2n+1,q)$ ($q\equiv 3 \pmod 4$) it is well known that S is isomorphic to a Sylow 2-subgroup of $GL(2n,q)$. Its rank and normal rank are given in section 3.1.

5.3. **Symplectic groups.** Let T be a generalised quaternion group whose order is the largest power of 2 that divides q^2-1 and let w_m be the wreath product of m copies of $\mathbb{Z}_2$. Then $T\wr w_m$ is a Sylow 2-subgroup of $Sp_{2^m}(q)$. Let $n=2^{m_1}+2^{m_2}+\cdots+2^{m_u}$ be the 2-adic representation of n and let T_i be a Sylow 2-subgroup of $Sp_{2n_i}(q)$ ($n_i=2^{m_i}$). By Theorem 6 of [W], $S=T_1\times T_2\times\cdots\times T_u$ is a Sylow 2-subgroup of $Sp_{2n}(q)$. We may apply Proposition 3.4 to conclude that the rank as well as the normal rank of S is n.

5.4. **Orthogonal Commutator Groups** $\Omega_{2n+1}(q)$**.** Let 2^{t+1} be the largest power of 2 that divides q^2-1 and let $T=\langle v,w\rangle$ be a dihedral group of order 2^t, where $o(v)=2^{t-1}, o(w)=2, wvw=v^{-1}$. Further, $R=\langle e\rangle$ is a group of order 2 acts on T by $o(e)=2$, $eve=v^{-1}, ewe=vw$. Let $n=2^{m_1}+2^{m_2}+\cdots+2^{m_u}$ be the 2-adic representation of n. Then $S(T,R,J)$ is a Sylow 2-subgroup of $\Omega_{2n+1}(q)$ (see (ii) of Theorem 7 of Wong [W]).

Case 1. T **is nonabelian.**

(a) **Normal rank of** $S(T,R,J)$**.** Direct calculation shows that de ($d\in T, e\in R$) is an involution if and only if $ede=d^{-1}$. As a consequence, $d=v^i$ for some i. Hence involutions in TR take the forms v^ie. One sees easily that if $x=v^ie\in TR-T$, then $\langle x,wxw^{-1}\rangle$ is not abelian. By (ii) of Proposition 4.4,

$$nr_2(S(T,R,J)=n\cdot nr_2(TR)=n.$$

Remark. In the case $|T|\geq 16$, the normal rank of T is 1. It follows that $nr_2(S)=n$. In the case $|T|=8$, the normal rank of T is 2. However, such normal subgroups can not be normalised by e. Hence $nr_2(TR)=1$.

(b) **Rank of** $S(T,R,J)$**.** We shall now determine $r_2(S)$. Recall that $de\in TR$ is of order 2 only if $d=v^i$. It follows that $r_2(TR)=r_2(T)=2$. We may apply (ii) of Proposition 4.9 and conclude that

$$r_2(S(T,R,J))=2n.$$

Case 2. T **is abelian.** This implies that $T \cong E_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consequently, we have $o(v) = o(w) = o(e) = 2$, $vw = wv$, $ev = ve$, $ewe = vw$. It follows that $S(T, R, J)$ is isomorphic to a Sylow 2-subgroup of the alternating group A_{4n}. It is well known that the rank of A_{4n} is $2n$ (see Proposition 5.2.10 of [GLS3]). Hence $S(T, R, J)$ is of rank $2n$ as well. Note that $S(T, 1, 1)$ is elementary abelian normal of rank $2n$. Hence the normal rank of $S(T, R, J)$ is $2n$.

5.5. **Orthogonal Commutator Groups** $\Omega_{2n}(\eta, q)$ **where** $\eta = \pm 1$**,** $q^n \equiv -\eta$ **(mod 4).** Applying Theorem 7 of Wong [W], a Sylow 2-subgroup of $\Omega_{2n}(\eta, q) = P\Omega_{2n}(\eta, q)$ is isomorphic to a Sylow 2-subgroup of $O_{2(n-1)}(\eta', q)$, where $q^{n-1} \equiv \eta'$ (mod 4). Let S be a Sylow 2-subgroup of $O_{2(n-1)}(\eta', q)$, where $q^{n-1} \equiv \eta'$ (mod 4). Applying Theorem 3 of Carter and Fong [CF], S is isomorphic to a Sylow 2-subgroup of $O^+_{2n-1}(q)$. We shall now describe S as follows :

Let D be a dihedral group of order 2^{s+1}, where 2^{s+1} is the largest power of 2 that divides $q^2 - 1$. Then D is isomorphic to a Sylow 2-subgroup of $O^+_3(q)$. Let T_{r-1} be the wreath product of $r - 1$ copies of $\mathbb{Z}_2$ and let S_r be the wreath product of D and T_{r-1}. Then S_r is a Sylow 2-subgroup of $O^+_{2^r+1}(q)$. Let $2(n - 1) = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_u}$ be the 2-adic representation of $2(n - 1)$. Applying Theorem 2 of Carter and Fong [CF],

$$S \cong S_{m_1} \times S_{m_2} \times \cdots \times S_{m_u}.$$

Note that the normal rank of D is 1 if D is of order 16 or more and 2 if the order of D is 8. The normal rank of S can be determined by applying Proposition 3.4. In particular, one has

(i) if $|D| \geq 16$, then $nr_2(S) = n - 1$,
(i) if $|D| = 8$, then $nr_2(S) = 2n - 2$

By Proposition 3.4, the rank of S is $2n - 2$.

5.6. **Orthogonal Commutator Groups** $\Omega_{2n}(\eta, q)$**, where** n **is even,** $\eta = \pm 1$**,** $q^n \equiv \eta$ **(mod 4).** Note that $\eta = 1$ since n is even and $q^n \equiv \eta$. Let 2^{t+1} be the greatest power of 2 that divides $q^2 - 1$ and let T be the central product of two dihedral groups of order 2^{t+1} :

$$T = \left\langle d = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, g = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} \right\rangle,$$

where $o(u) = 2^t$, $o(w) = 2$, $wuw = u^{-1}$. Let $R = \langle e, f \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where

$$d^e = g^{-1}, g^e = d^{-1}, h^e = gk, k^e = dh,$$
$$d^f = g, g^f = d, h^f = k, k^f = h.$$

Let $n/2 = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_u}$ (n even) be the 2-adic representation of $n/2$. By Theorem 11 of [W], $S(T, R, J)$ is a Sylow 2-subgroup of $\Omega_{2n}(\eta, q)$, where n is even, $\eta = \pm 1$, $q^n \equiv \eta$ (mod 4).

(a) **Rank of** $S(T, R, J)$**.** One sees easily that $r_2(TR) = 4 = r_2(T) + 1$. Let $E_0 = \langle dg, d^{-1}g, hk \rangle$, $R_0 = \langle f \rangle$. Then $r_2(E_0) = 3$, $[E_0, R_0] = 1$. Further, $S(E_0, R_0, 1)$ is of rank $2n - 1$. Applying (iii) of Proposition 4.9,

$$r_2(S(T, R, J)) = (n/2)r_2(TR) - 1 = (n/2)4 - 1 = 2n - 1.$$

Remark. Since $[E_0, R_0] = 1$, $[S(E_0, R_0, 1), \mathrm{diag}\,(f, 1, \cdots)] = 1$.

(b) **Normal Rank of** $S(T, R, J)$**.** We shall now study the normal rank of $S(T, R, J)$.

b(i) **Case 1.** $t \geq 3$**.** Direct calculation shows that $v = d^x h^y g^z k^w e$ is of order 2 if and only if

(i) $y \equiv w \equiv 0 \pmod 2$, $x \equiv z \pmod{2^{t-1}}$, or

(ii) $y \equiv w \equiv 1 \pmod 2$, $x + z - 1 \equiv 0 \pmod{2^{t-1}}$.

Further, $\langle v, dvd^{-1} \rangle$ is not abelian.

In the case $r = d^x h^y g^z k^w f$, r is of order 2 if and only if

(i) $y \equiv w \equiv 0 \pmod 2$, $x + z \equiv 0 \pmod{2^{t-1}}$, or

(ii) $y \equiv w \equiv 1 \pmod 2$, $x - z \equiv 0 \pmod{2^{t-1}}$.

Further, $\langle r, drd^{-1} \rangle$ is not abelian.

In the case $s = d^x h^y g^z k^w ef$, s is of order 2 if and only if $y \equiv w \equiv 0 \pmod 2$. Further, $\langle s, dsd^{-1} \rangle$ is not abelian.

By (ii) of Proposition 4.4, $nr_2(S(T, R, J)) = n \cdot nr_2(TR)$. It is easy to see that $nr_2(TR) = 2$ and that

$$E_0 = \left\langle (dg)^{2^{t-2}}, (d^{-1}g)^{2^{t-2}} \right\rangle \lhd TR$$

is of rank 2. Hence $nr_2(S(T, R, J)) = (n/2)nr_2(TR) = n$. Further, $S(E_0, 1, 1)$ is elementary abelian normal of dimension n.

Remark. Note that our results about elements of order 2 in $TR - T$ works for $t = 2$ as well.

b(ii) **Case 2.** $t = 2$**.** In this case, T is a central product of two dihedral groups of order 8. Note first that $nr_2(T) = 3$. Let $E_0 = \langle dg, d^{-1}g, hk \rangle \lhd T$, $R_0 = \langle f \rangle \lhd R$. Then $[E_0, R_0] = 1$ and E_0R_0 is normal elementary abelian of rank 3 in TR. It follows that $S(E_0, R_0, 1)$ is normal elementary abelian in $S(T, R, J)$ and that

$$r_2(S(E_0, R_0, 1)) = (n/2)r_2(E_0) + (n/2 - 1)r_2(R_0) = 2n - 1.$$

As a consequence, $nr_2(S) = 2n - 1$ ((i) of Proposition 4.4).

Remark. Since $[E_0, R_0] = 1$, we have $[S(E_0, R_0, 1), \mathrm{diag}\,(f, 1, \cdots)] = 1$.

5.7. **Orthogonal Commutator Groups** $\Omega_{2n}(\eta, q)$ **where** n **is odd,** $\eta = \pm 1$**,** $q^n \equiv \eta$ **(mod 4).** Let $n = 1 + n_1$, where n_1 is even. Let S be a Sylow 2-subgroup of $\Omega_{2n_1}(\eta, q)$ where n_1 is even, $\eta = \pm 1$, $q^{n_1} \equiv \eta \pmod 4$ and let $\mathrm{diag}\,(e, 1, 1, \cdots, 1) = d(e)$, $\mathrm{diag}\,(f, 1, 1, \cdots, 1) = d(f)$ (e and f are given in section 5.6). Set

$$\langle S, d(e), d(f) \rangle \times \langle x, y \rangle, \text{ where } o(x) = o(y) = 2, o(xy) = 2^t.$$

Let $D = \langle d(e)x, d(f)y \rangle$ (dihedral of order 2^{t+1}). By Theorem 12 of Wong [W],

$$V = \langle S, d(e)x, d(f)y \rangle = S \rtimes D \tag{5.1}$$

is a Sylow 2-subgroup of $\Omega_{2n}(\eta, q)$.

(a) **Normal Rank of** V**.** Let E be normal elementary abelian of dimension $nr_2(V)$. Recall that $D = \langle d(e)x, d(f)y \rangle$ is dihedral of order 2^{t+1} ($t \geq 2$). Set

$$E = (E \cap S) \times B. \tag{5.2}$$

By Lemma 2.3, $r_2(B) \leq 2$. Hence

$$nr_2(V) = r_2(E) \leq nr_2(S) + 2. \tag{5.3}$$

In the case D is of order 8 ($t = 2$), D has a normal subgroup $\langle d(f)y, (xy)^2 \rangle = D_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $K = S(E_0, R_0, 1) \subseteq S$ be given as in b(ii) of section 5.6. Then K is elementary abelian of dimension $nr_2(S)$. Since $\langle x, y \rangle$ commutes with S and $[K, d(f)] = 1$ (see remark of b(ii) of section 5.6), $[K, D_0] = 1$. Hence $K \times D_0$ is

normal elementary abelian (of V) of dimension $nr_2(S)+2$. Applying inequality (5.3), we have

$$nr_2(V) = nr_2(S) + 2 = (2n_1 - 1) + 2 = 2n_1 + 1 = 2n - 1.$$

In the case, $|D| > 8$ ($t \geq 3$). If E is of dimension $nr_2(S)+2$. Then B (see (5.2)) is of dimension 2. Consequently, $B|D$ (see Lemma 2.2 for the notation $B|D$) is elementary abelian of dimension 2. Since E is normal in $V = S \rtimes D$, one can show that $E|D = B|D$ (of dimension 2) is normal in D. This is a contradiction (the normal rank of D is 1). Hence (5.3) can be improved to

$$nr_2(V) \leq nr_2(S) + 1 = n_1 + 1. \tag{5.4}$$

Let $K = S(E_0, 1, 1) \subseteq S$ be normal elementary abelian (of S) of dimension $nr_2(S)$ (see b(i) of section 5.6) and let $D_0 = \langle (xy)^{o(xy)/2} \rangle = Z(D)$. It is clear that $K \times D_0$ ($K \subseteq S$ commutes with xy) is normal elementary abelian (of V) of dimension $r_2(S)+1$. Applying inequality (5.4), we have

$$nr_2(V) = nr_2(S) + 1 = n_1 + 1 = n.$$

In summary, we have

(i) if $t \geq 3$, then $nr_2(V) = n_1 + 1 = n$,
(ii) if $t = 2$, then $nr_2(V) = 2n_1 + 1 = 2n - 1$.

(b) **Rank of V.** We shall now determine the rank of $V = S \rtimes D$. Let E be elementary abelian of $\langle S, d(e)x, d(f)y \rangle$ of dimension $r_2(V)$. Let

$$E = A \times B, \text{ where } A = E \cap S.$$

Since $r_2(B) \leq r_2(D)$ (Lemma 2.3) and $r_2(S) = 2n_1 - 1$ (see (a) of section 5.6), we have

$$r_2(V) = r_2(E) = r_2(A) + r_2(B) \leq r_2(S) + 2 = 2n_1 + 1.$$

Let $\langle d(f)y, (xy)^{o(xy)/2} \rangle = D_0 \subseteq D$. Then D_0 is elementary abelian of dimension 2. Let $K = S(E_0, R_0, 1) \subseteq S$ be elementary abelian of dimension $r_2(S) = 2n_1 - 1$ (see (a) of section 5.6). Applying the remark of (a) of section 5.6 and the fact that $[S, \langle x, y \rangle] = 1$, $KD_0 = K \times D_0$ is elementary abelian of dimension $2n_1 + 1$. Hence $r_2(V) = 2n_1 + 1 = 2n - 1$.

References

[CF] R. Carter, P. Fong, *The Sylow 2-subgroups of the Finite Classical Groups*, J. of Algebra **1** (1964), 139-151.

[GLS2] G. Gorenstein, R. Lyons, R. Solomon, *The Classification of the Finite Simple Groups, Number* 2, Amer. Math. Soc. Providence, Rhode Island (1998).

[W] W. J. Wong, *Twisted Wreath Products and Sylow 2-subgroups of Classical Simple Groups*, Math. Z. **97** (1967), 406-424.

DEPARTMENT OF MATHEMATICS,
NATIONAL UNIVERSITY OF SINGAPORE,
SINGAPORE 117543,
REPUBLIC OF SINGAPORE.

e-mail: matlml@math.nus.edu.sg

lang-67-6.tex